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CMS Technical Summary Report #90-5

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# STABILITY OF DISCONTINUOUS STEADY STATES IN SHEARING MOTIONS OF A NON-NEWTONIAN FLUID*


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## Abstract

This document studies ~~We study~~ the nonlinear stability of discontinuous steady states of a model initial-boundary value problem in one space dimension for incompressible, isothermal shear flow of a non-Newtonian fluid driven by a constant pressure gradient. The non-Newtonian contribution to the shear stress is assumed to satisfy a simple differential constitutive law. The key feature is a non-monotone relation between the total steady shear stress and shear strain-rate that results in steady states having, in general, discontinuities in the strain rate. ~~We~~ show that every solution tends to a steady state as $t \to \infty$, and we identify steady states that are stable.



AMS (MOS) Subject Classifications: 34D20, 35B35, 35B45, 35B65, 35F25, 73F15, 76A10

Key Words: stability, discontinuous steady states, invariant regions, well-posedness, asymptotic behavior, non-Newtonian fluids



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## 1. Formulation and Discussion of Model Problem

We study the quasilinear system

$$v_t = S_x \;, \tag{1.1}$$

$$\sigma_t + \sigma = g(v_x) \;, \tag{1.2}$$

on $[0,1] \times [0,\infty)$, where

$$S := T + f\,x, \qquad T := \sigma + v_x \tag{1.3}$$

with $f$ a fixed positive constant throughout. We impose the boundary conditions

$$S(0,t) = 0 \quad \text{and} \quad v(1,t) = 0, \qquad t \geq 0\,, \tag{1.4}$$

and the initial conditions

$$v(x,0) = v_0(x), \quad \sigma(x,0) = \sigma_0(x), \qquad 0 \leq x \leq 1\,; \tag{1.5}$$

accordingly,

$$S(x,0) = S_0(x) := \sigma_0(x) + v_{0x}(x) + fx\,. \tag{1.6}$$

The function $g : \mathbb{R} \to \mathbb{R}$ is assumed to be smooth, odd, and $\xi\, g(\xi) > 0$, for $\xi \neq 0$.

The initial-boundary value problem (1.1)–(1.5) serves as a simple model for studying the dynamic behavior of a non-Newtonian fluid in Poiseuille shear flow between parallel plates located at $x = \pm 1$; the flow is driven by a *constant pressure gradient* $f$. In this context, $v$, the velocity of the fluid in the channel, and $T$, the shear stress, are connected through the balance of linear momentum (1.1). The shear stress $T$ is decomposed into a non-Newtonian *contribution* $\sigma$, *evolving* in accordance with the differential constitutive law (1.2), and a viscous contribution $v_x$. The flow is assumed to be symmetric about the centerline of the channel. Symmetry and compatibility with boundary conditions dictate the following *restrictions* on the initial data:

$$v_0(1) = 0, \qquad S_0(0) = 0, \quad \text{and} \quad \sigma_0(0) = 0\,, \tag{1.7}$$

which, together with Eqs. (1.2)–(1.4), guarantee that

$$\sigma(0,t) = v_x(0,t) = 0\,, \tag{1.8}$$

and symmetry is preserved for all time.

The system (1.1)–(1.4) admits steady state solutions $(\overline{v}(x), \overline{\sigma}(x))$ satisfying

$$\bar{S} := g(\bar{v}_x) + \bar{v}_x + fx = 0, \qquad \bar{\sigma} = g(\bar{v}_x) \tag{1.9}$$

on the interval $[0,1]$. In case the function $w(\xi) := g(\xi) + \xi$ is *not monotone*, there may be multiple values of $\overline{v}_x(x)$ that solve Eq. (1.9) for some $x's$, thus leading to steady velocity profiles with kinks (see Figs. 1 and 2). Our objective is to study the stability of such

velocity profiles; we also study well-posedness and the convergence of solutions of (1.1)–(1.5) to steady states as $t \to \infty$.

Problem (1.1)–(1.5) captures certain key features of a class of viscoelastic models that have been proposed to explain the occurence of "spurt" phenomena in non-Newtonian flows. The phenomenon of spurt was apparently first observed by Vinogradov *et al.* [18] in the flow of highly elastic and very viscous non-Newtonian fluids through capillaries or slit dies. It is associated with a sudden increase of the volumetric flow rate occuring at a critical stress that appears to be independent of the molecular weight. It has been proposed by Hunter and Slemrod [5] and Malkus, Nohel, and Plohr [8], [9], and [10] that spurt phenomena may be explained by constitutive laws that lead to a nonmonotone relation of the total steady shear stress versus the shear strain-rate. In this framework, the increase of the volumetric flow rate corresponds to jumps in the strain rate when the driving pressure gradient exceeds a critical value.

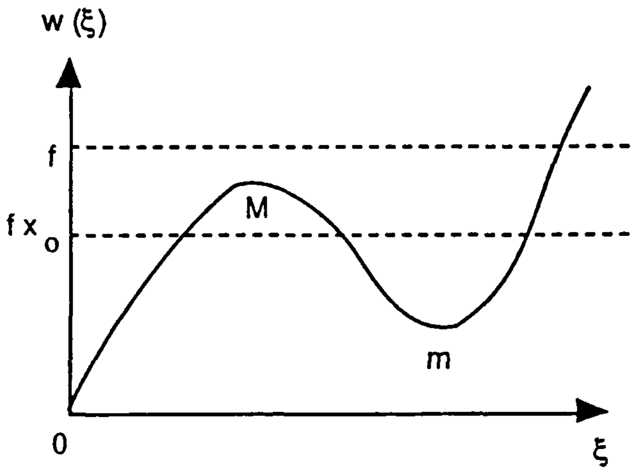


**Fig. 1**: $w$ *vs.* $\xi$.

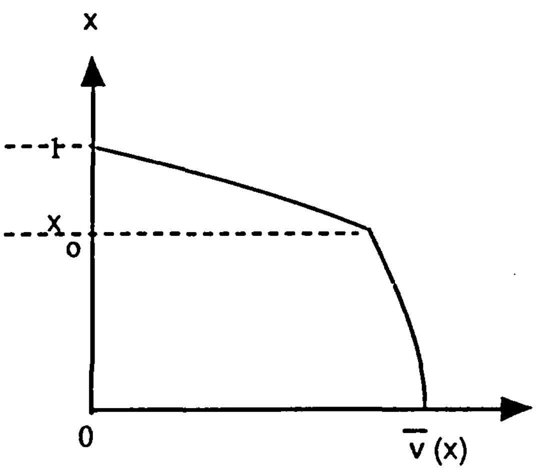


**Fig. 2**: Velocity profile with a kink; $w(-\bar{v}_x(x)) = fx$.

Interestingly, various three-dimensional differential constitutive laws, when restricted to one-dimensional shearing motions, produce nonmonotone steady shear stress vs. steady shear strain-rate relations. The simplest such model leads to the initial-boundary value problem

$$\begin{aligned} \alpha v_t &= \sigma_x + \varepsilon v_{xx} + f \ , \\ \sigma_t + \sigma &= (Z+1)\, v_x \ , \\ Z_t + Z &= -\sigma v_x \ . \end{aligned} \tag{1.10}$$

with boundary conditions (1.4) and appropriate initial conditions; the parameters $\alpha$ and $\varepsilon$ represent the ratio of Reynolds number to Deborah number and a ratio of viscosities, respectively. The evolution of $\sigma$, the polymer contribution to the shear stress, and of $Z$, a quantity proportional to the normal stress difference, are governed by the second and third equations in (1.10), which are a restriction of differential constitutive laws due to Oldroyd [12] and Johnson and Segalman [6] (see [16] or [8] for a derivation). The steady

state solutions $(\overline{v}(x), \overline{\sigma}(x), \overline{Z}(x))$ satisfy

$$\overline{\sigma} = \frac{\overline{v}_x}{1+\overline{v}_x^2}, \qquad \overline{Z}+1 = \frac{1}{1+\overline{v}_x^2}, \tag{1.11}$$

and the steady strain-rate $\overline{v}_x$ is determined by solving the equation

$$\overline{T}(\overline{v}_x) := \frac{\overline{v}_x}{1+\overline{v}_x^2} + \varepsilon \overline{v}_x = -f\,x\ . \tag{1.12}$$

For $\varepsilon < \frac{1}{8}$, $\overline{T}(\xi)$ is not monotone; its graph has the shape shown in Fig. 1. Malkus, Nohel and Plohr [8] performed numerical calculations for solutions of (1.10) that reproduce the experimental results on spurt of Vinogradov *et al.* [18] qualitatively and quantitatively. Moreover, motivated by the relative scales of the parameters $\alpha$ and $\varepsilon$ in the experiments, they study an associated system of ordinary differential equations that is obtained through a "quasidynamic approximation" (setting $\alpha = 0$) in (1.10). Their analysis of the approximating system in [9] and [10] predicts spurt as well as latency, shape memory and hysteresis under cyclic loading.

Here, we test the hypothesis that lack of monotonicity of the steady shear stress function causes non-Newtonian response, at the level of the paradigm (1.1)–(1.5). We remark that this problem can be regarded as a formal approximation to (1.10) arising by fixing $Z$ in $(1.10_2)$ at its steady state value in (1.11). For simplicity, the function $w(\xi) = g(\xi) + \xi$ is assumed to have one single loop. The loop forms as the combined outcome of the non-Newtonian contribution to the steady shear stress associated with $g(\xi)$ and the Newtonian contribution associated with $\xi$. The hypotheses on $g(\xi)$ imply that $w(\xi)$ is odd and $w(\xi) \neq 0$ for $\xi \neq 0$. The graph of a representative $w(\xi)$ for $\xi > 0$ is shown in Fig. 1; in the figure, $m$ and $M$ stand for the levels of the bottom and top of the loop, respectively. Our analysis can be routinedly generalized to cover the case when $w(\xi)$ has a finite number of loops, but we do not pursue this direction here.

Steady state solutions for the representative $w(\xi)$ are constructed as follows: First, solve $w(\overline{u}(x)) = fx$ for each given $x \in [0,1]$, where $\overline{u}(x) = -\overline{v}_x(x)$. This equation admits a unique solution for $0 < fx < m$ or $fx > M$, and three solutions for $m < fx < M$. Let $\overline{u}(x)$, $0 \leq x \leq 1$, be such a solution. Setting

$$\overline{v}(x) = \int_x^1 \overline{u}(y)dy, \qquad \overline{\sigma}(x) = g(-\overline{u}(x)) \tag{1.13}$$

$(\overline{v}(x), \overline{\sigma}(x))$ satisfy (1.9) and (1.4) and give rise to a steady state. Clearly, if $f < m$ there is a unique smooth steady state, if $m < f < M$ there is a unique smooth steady velocity profile and a multitude of profiles with kinks, finally, if $f > M$ all steady velocity profiles have kinks. An example of a velocity profile with a kink is shown in Fig. 2.

We are interested in studying the dynamics of solutions to (1.1)–(1.5), and in particular, to determine which of the steady states are stable. We establish that every solution of (1.1)–(1.5) converges as $t \to \infty$ to a steady state solution $(\overline{v}(x), \overline{\sigma}(x))$. It would be interesting to identify the region of attraction of the stable steady solutions, however, this

seems a very complicated task. Instead, we show that *steady states* $(\overline{v}(x), \overline{\sigma}(x))$, *with a* finite number of jumps in $\overline{v}_x(x)$ or $\overline{\sigma}(x)$, and such that

$$w'(-\overline{v}_x(x)) \geq c_0 > 0, \qquad a.e. \quad x \in [0,1] \tag{1.14}$$

are "asymptotically stable" in a sense that is made precise in Theorem 5.1. Note that relation (1.14) excludes solutions that take values in the monotone nonincreasing part of the graph of $w(\xi)$. The stable solutions are local minimizers of an associated energy functional.

The above results are close, in spirit and in technique, to the analysis of Andrews and Ball [1] and, especially, that of Pego [14] for phase transitions in one-dimensional viscoelastic materials of rate type. Current work of Novick-Cohen and Pego [11] on spinodal decomposition involves similar ideas.

The paper is organized as follows: In Sec. 2, we obtain basic a-priori estimates and determine invariant regions for an associated ordinary differential equation. In Sec. 3, we discuss the well-posedness and regularity of solutions of a system incorporating (1.1)–(1.5), a consequence of which are existence and regularity results for (1.10) and other popular models for motions of non-Newtonian fluids arising from differential constitutive laws. In Sec. 4, we study the convergence of classical solutions of (1.1)–(1.5) to steady states. Finally, in Sec. 5, we state and prove the main result on stability of steady states by making use of the analysis in Secs. 2 and 4.

## 2. A Priori Estimates and Invariant Regions

Problem (1.1)–(1.5) is endowed with certain identities that play a crucial role in the analysis. As a consequence of (1.1) and (1.4), smooth solutions of the system (1.1)–(1.5) satisfy:

$$1/2\frac{d}{dt}\int_0^1 v^2 dx + \int_0^1 S v_x dx = 0\,, \tag{2.1}$$

and

$$1/2\frac{d}{dt}\int_0^1 v_t^2 dx + \int_0^1 v_t^2 dx + \int_0^1 (S_t + S) v_{xt} dx = 0\,. \tag{2.2}$$

Using (1.3) and (1.2), Eq. (2.2) leads to the energy identity

$$\frac{d}{dt}\{1/2\int_0^1 v_t^2 dx + \int_0^1 [W(v_x) + x\, f\, v_x] dx\} + \int_0^1 [v_t^2 + v_{xt}^2] dx = 0\ . \tag{2.3}$$

The function

$$W(\xi) := \int_0^\xi w(\zeta) d\zeta = 1/2\xi^2 + \int_0^\xi g(\zeta) d\zeta \tag{2.4}$$

plays the role of a stored energy function, and is not convex. This fact is the main obstacle in the analysis of stability. Since $\xi g(\xi) > 0$, it follows that $\int_0^\xi g(\zeta)d\zeta \geq 0$ for $\xi \in \mathbb{R}$, and $W(\xi)$ satisfies the lower bound

$$W(\xi) + f x \xi \geq 1/4\xi^2 - f^2\,, \qquad \xi \in \mathbb{R}, 0 \leq x \leq 1\,. \tag{2.5}$$

Integrating (2.3), one has the identity

$$1/2\int_0^1 v_t^2(x,t)dx + \int_0^1 (W(v_x(x,t)) + fxv_x(x,t))dx + \int_\tau^t \int_0^1 (v_t^2 + v_{xt}^2)dxds =$$
$$1/2\int_0^1 v_t^2(x,\tau)dx + \int_0^1 (W(v_x(x,\tau) + fxv_x(x,\tau))dx, \qquad 0 \le \tau \le t. \tag{2.6}$$

As a first consequence, the energy identity (2.6) yields an a priori bound for $S$ as follows. First observe that (2.5) and (2.6) with $\tau = 0$ imply the estimate

$$\int_0^1 v_t^2(x,t)dx + 1/2\int_0^1 v_x^2(x,t)dx + 2\int_0^t \int_0^1 (v_t^2 + v_{xt}^2)dxds$$
$$\le 2f^2 + \int_0^1 S_{0x}^2(x)dx + 2\int_0^1 [W(v_{0x}(x)) + fxv_{0x}(x)]dx \le C\,, \tag{2.7}$$

where $C$ is a constant depending only on the data. From (1.1) and (1.4) we obtain the inequality

$$S^2(x,t) = [\int_0^x v_t(\xi,t)d\xi]^2 \le \int_0^1 v_t^2(x,t)dx\,, \tag{2.8}$$

which, together with (2.7), implies

$$|S(x,t)| \le C \qquad 0 \le x \le 1, \quad 0 \le t < \infty. \tag{2.9}$$

Control of $S$ enables us to take advantage of the special structure of Eq. (1.2). It is convenient to introduce the quantity

$$s := \sigma + f\,x\,. \tag{2.10}$$

Then, Eqs. (1.2), (1.3), and (2.10) readily imply

$$s_t + s + g(s - S) = f\,x. \tag{2.11}$$

For a fixed $x$, Eq. (2.11) may be viewed as an ODE with the forcing term $S(x,\bullet)$. Also, observe that for a steady state $(\overline{\sigma},\overline{v}_x)$, one has $\overline{S} = 0$, and $\overline{s} = -\overline{v}_x$ is an equilibrium solution of (2.11) (with $S = 0$). If $S \equiv 0$ in (2.11), it is evident that the ODE admits positively invariant intervals for each fixed $x$. We claim that this property is preserved in the presence of a priori control of $S$. Such control is provided by bounding the $L^2$-norm of $v_t$ in (2.8) as in (2.9); more delicate bounds are derived in Sec. 5.

To fix ideas, assume that for $0 \le t \le t_0$

$$|S(x,t)| \le \rho \quad , \quad 0 \le x \le 1 \tag{2.12}$$

for some $\rho > 0$. For $x$ fixed in $[0,1]$, we use the notation $S(t) := S(x,t)$ and conveniently rewrite (2.11) as

$$s_t + w(s - S(t)) = f\,x - S(t)\,. \tag{2.13}$$

We state the following lemma on invariant intervals; its proof is obvious.

**Lemma 2.1.** *Let $S$ satisfy the uniform bound* (2.12) *for* $0 \le t \le t_0$. *For $x$ fixed,* $0 \le x \le 1$, *assume there exist $s_-$, $s_+$ such that $s_- < s_+$ and*

$$w(s_- - \lambda) < f\,x - \lambda \quad , \quad |\lambda| \le \rho \tag{2.14}$$

$$w(s_+ - \lambda) > f\,x - \lambda \quad , \quad |\lambda| \le \rho \tag{2.15}$$

*Then the compact interval $[s_-, s_+]$ is positively invariant for the ODE* (2.13) *on the time interval* $0 \le t \le t_0$.

With the goal of generating invariant intervals, we study the solution sets of the inequalities (2.14) and (2.15) as functions of $\rho$ and $x$. First observe that, since $\lim_{\xi \to \pm\infty} w(\xi) = \pm\infty$, given any $x$ and $\rho$ one can determine $s_{0+}$ large, positive and $s_{0-}$ large, negative such that if $s_- < s_{0-}$ and $s_+ > s_{0+}$, then $s_-$ and $s_+$ satisfy (2.14) and (2.15), respectively.

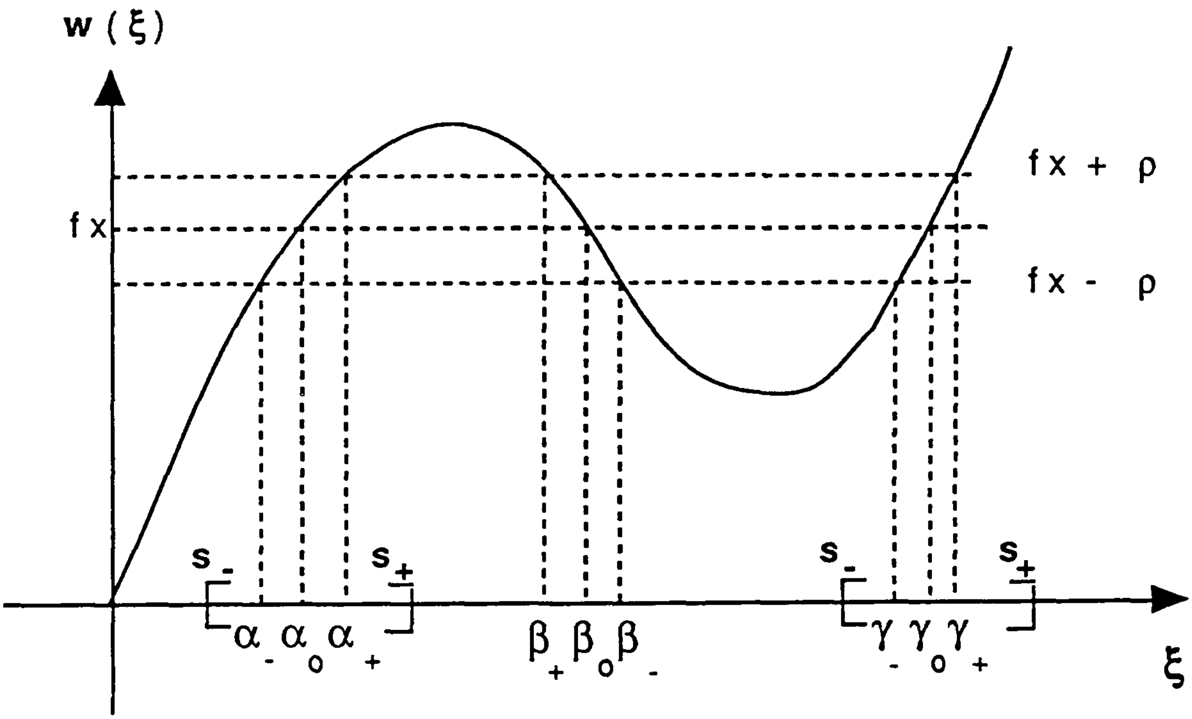


**Fig. 3:** Invariant Intervals.

The possibility for more discriminating choices of invariant intervals arises if we restrict attention to small values of $\rho$. For a function $w(\xi)$ with a single loop, the most interesting case arises when $f\,x - \rho$, $f\,x$ and $f\,x + \rho$ each intersects the graph of $w(\xi)$ at three distinct points. Referring to Fig. 3, the abscissae of the points of intersection are denoted by $(\alpha_-, \beta_-, \gamma_-)$, $(\alpha_0, \beta_0, \gamma_0)$ and $(\alpha_+, \beta_+, \gamma_+)$, respectively. First examine (2.15). For $0 < \lambda < \rho$, any $s_+ \in (\alpha_0 + \rho, \beta_0) \cup (\gamma_0 + \rho, \infty)$ satisfies (2.15), while, for $-\rho < \lambda < 0$, any $s_+ \in (\alpha_+, \beta_+ - \rho) \cup (\gamma_+, \infty)$ satisfies (2.15) (here, by convention, any interval that is not well defined will stand for the empty set). Thus, the solution set of (2.15) contains an unbounded component $J_u := (\gamma_0 + \rho, \infty) \cap (\gamma_+, \infty)$, as well as a bounded component

$J_b := (\alpha_0+\rho, \beta_0) \cap (\alpha_+, \beta_+ - \rho)$. Correspondingly, the solution set of (2.14) also contains two components $I_u := (-\infty, \alpha_-) \cap (-\infty, \alpha_0 - \rho)$ and $I_b := (\beta_0, \gamma_0 - \rho) \cap (\beta_- + \rho, \gamma_-)$. Choosing $s_- \in I_u$ and $s_+ \in J_u$ we construct invariant intervals $[s_-, s_+]$ of the type discussed in the previous paragraph. More interesting choices occur if one chooses $\rho$ so small that $I_b$ and/or $J_b$ are nonempty; one then takes $s_- \in I_b$ and $s_+ \in J_u$, or $s_- \in I_u$ and $s_+ \in J_b$ (see Fig. 3).

The above discussion identifies invariant intervals for (2.13) when $x$ is fixed. Of particular interest to us is to study the persistence and dependence on $x$ of the invariant intervals as $x$ varies in $[0, 1]$. In this context $\rho$ is fixed and (2.12) is assumed to hold. $I_u$, $J_u$, $I_b$ and $J_b$ are defined as above and depend on $x$ as well as $\rho$.

If we choose $s_- \in \bigcap_{x\in[0,1]} I_u$ and $s_+ \in \bigcap_{x\in[0,1]} J_u$ (which are nonempty) then $[s_-, s_+]$ is an invariant interval for (2.13) valid for all $x \in [0, 1]$. As an application of this remark and by virtue of the bound (2.9), we conclude that

$$|s(x,t)| \leq C\,, \qquad 0 \leq x \leq 1,\, t \geq 0 \tag{2.16}$$

which, in turn, using (1.3) and (2.10), implies

$$|v_x(x,t)| \leq C\,, \qquad 0 \leq x \leq 1,\, t \geq 0\,. \tag{2.17}$$

By contrast, the discriminating invariant intervals degenerate as we approach the top or bottom of the loop. However, if $\rho$ is small enough and we avoid the top or bottom of the loop, the invariant intervals may be chosen of uniform length. In Sec. 5 we use a result along these lines which we formulate below.

Let $\overline{s}(x)$ be a piecewise smooth solution of

$$w(\overline{s}(x)) = fx \tag{2.18}$$

defined on $[0, 1]$ and admitting jump discontinuities at a finite number of points $x_1, ..., x_n$ in $[0, 1]$. In addition. suppose that $\overline{s}(x)$ takes values in the monotone increasing parts of the curve $w(\xi)$ and that it avoids jumping at the top or bottom of the loop, i.e.,

$$w'(\overline{s}(x)) \geq c_0 > 0\,, \qquad x \in [0,1] \setminus \{x_1, ..., x_n\}\,, \tag{2.19}$$

for some constant $c_0$. We look for invariant intervals for (2.13) that are of uniform length, centered around $\overline{s}(x)$ at each $x$. First, consider the case $\rho = 0$ in (2.12), i.e. $S \equiv 0$, in (2.13). For each fixed $x$, let $R_0(x)$ be the distance between $\overline{s}(x)$ and the nearest solution of the equation $w(\xi) = fx$ different from $\overline{s}(x)$. Set $R_0 := \inf_{0 \leq x \leq 1} R_0(x)$ and observe that by (2.19), $R_0 > 0$. Then for any $r < R_0$, $[\overline{s}(x) - r, \overline{s}(x) + r]$ is an invariant interval for (2.13) with $S \equiv 0$. For $\rho > 0$, a review of the above constructions yields:

**Corollary 2.2.** *Given $\overline{s}(x)$ satisfying (2.18) and (2.19), there are positive constants $\rho_0$ and $R_0$, such that given any $\rho < \rho_0$ there are positive numbers $r_1(\rho)$ and $r_2(\rho) = R_0 - r_1(\rho)$ with the property: If (2.12) holds on $[0, t_0]$, then for any $r \in (r_1(\rho), r_2(\rho))$, the interval $[\overline{s}(x) - r, \overline{s}(x) + r]$ is positively invariant for the ode (2.13) on $[0, t_0]$ for any $x \in [0, 1] \setminus \{x_1, ..., x_n\}$. The function $r_1(\rho)$ is an increasing function of $\rho$, which is of the order of $\rho$, as $\rho \to 0$.*

## 3. Existence and Regularity Theory

In this section, we study existence and regularity questions for a coupled system of partial differential equations that describes one-dimensional shearing motions of non-Newtonian fluids. This system is appropriate for non-Newtonian fluid models in which the total stress is decomposed into a Newtonian contribution and a finite number of relaxation stress components, viewed as internal variables and evolving in accordance with differential constitutive laws. Some specific models include (1.10) and others that are discussed in the Appendix.

Let $v$ be a scalar and $U = (U_1, ..., U_N)$ be a N-vector and consider the system of partial differential equations

$$\begin{aligned} v_t &= S_x \, , \\ U_t &= H^*(U, v_x, x) \, , \\ S &= v_x + G^*(U, x) \, , \end{aligned} \tag{3.1}$$

on $[0,1] \times [0, \infty)$. In the context of viscoelasticity, $v$ is a scalar velocity and $U$ represents a vector of relaxation stresses viewed as internal variables. For definiteness, we impose boundary conditions appropriate for symmetric channel flow (some other boundary conditions may be treated in a similar fashion):

$$S(0,t) = 0 \, , \quad v(1,t) = 0, \qquad t \geq 0 \, , \tag{3.2}$$

and initial conditions of the form

$$v(x,0) = v_o(x) \, , \quad U(x,0) = U_o(x), \qquad x \in [0,1] \, . \tag{3.3}$$

We assume that the scalar function $G^* \in C^3$ and the vector function $H^* \in C^2$ with respect to their arguments for $U \in \mathbb{R}^N$, $v_x \in \mathbb{R}$, and $x \in [0,1]$. We indicate in the text whenever different smoothness hypotheses are imposed on the functions $G^*$ and $H^*$.

Two parallel existence and regularity theories will be pursued. The first is appropriate for smooth initial data and yields classical solutions. The second allows discontinuities in $v_{0x}(x)$ and in $U_0(x)$ and yields strong solutions. The latter result allows prescription of discontinuous initial data of the same type as the discontinuous steady states studied in this paper. Finally, we develop criteria for global existence which apply to the models at hand, namely, to systems (1.1)–(1.5), (1.10), as well as to some more general models discussed in the Appendix. We note that existence of classical solutions to the system (3.1)–(3.3) may also be obtained by using an approach based on the Leray - Schauder fixed point theorem (cf. Tzavaras [17] for existence results on a related system). Other existence results for (1.10) were obtained by Guillopé and Saut [2], and small-data results for some other models in more than one space dimension in [3].

Our first result establishes local-time existence of classical solutions of (3.1)–(3.3).

**Theorem 3.1** (Local classical solutions): *Assume that* $v_o(1) = 0$, *and that with* $S_o(x) = v_{ox}(x) + G^*(U_o(x), x)$, *we have* $S_o \in H^s[0,1]$ *for some* $s > 3/2$, *with* $S_o(0) = 0$, $v_0(1) = S_{ox}(1) = 0$, *and that* $U_o \in C^1[0,1]^N$, *where* $H^s$ *denotes the usual interpolation space* [7]. *Then for some* $T > 0$ *a unique solution of* (3.1)–(3.3) *exists with* $v \in C([0,T), C^2)$, $v_t \in$

$C([0,T),C^0)$, $U \in C^1([0,T],(C^1)^N)$. *If $T$ is the maximal time of existence, but $T$ is finite, then*

$$\limsup_{t \to T^-} [\sup_{x \in [0,1]} (|v_x(x,t)| + |U(x,t)|)] = \infty . \tag{3.4}$$

*Proof.* It is convenient to study the problem with $v$ replaced by $S$. After computing $S_t$ from (3.1) and substituting for $v_x = (S - G^*)$, we formally obtain equations of the form

$$S_t = S_{xx} + G(U,S,x) , \qquad U_t = H(U,S,x) \tag{3.5}$$

where $G$ and $H$ are $C^2$ functions. $S$ should satisfy the boundary conditions

$$S(0,t) = 0 , \qquad S_x(1,t) = 0 . \tag{3.6}$$

We study existence for the degenerate semilinear reaction-diffusion system in (3.5) using the standard approach of Henry [4]. We consider (3.5) as an abstract parabolic equation $z_t + Az = f(z)$ on a Banach space $X$. Here, $z = (S,U)$, the operator $A$ is defined by $A(S,U) = (-S_{xx}, 0)$ and the Banach space $X = L^2[0,1] \times C^1[0,1]^N$. The domain of the Laplacian $-\Delta S = -S_{xx}$ in $L^2$ is

$$D(-\Delta) = \{S \mid \quad S \in H^2[0,1] \quad \text{with} \quad S(0) = 0,\, S_x(1) = 0\} ,$$

so the domain of $A$ is $D(-\Delta) \times (C^1)^N$. The Laplacian is a sectorial operator in $L^2$, so $A$ is a sectorial operator on $X$, and generates an analytic semigroup. For $0 \le \alpha \le 1$, the domain of the fractional power $A^\alpha$ is $X^\alpha = D((-\Delta)^\alpha) \times (C^1)^N$. For $\alpha > 3/4$ this domain is continuously embedded into $C_b^1 \times (C^1)^N$, where

$$C_b^1 = \{S \in C^1[0,1] \mid \quad S(0) = 0,\, S_x(1) = 0\} .$$

The hypotheses on $G^*$ and $H^*$ imply that the map $(S,U) \to (G(U,S,x), H(U,S,x))$ is $C^1$ as a map from $X^\alpha$ to $X$. The hypotheses on the initial data imply that initially $z(0) \in X^\alpha = D(A^\alpha)$ for some $\alpha > 3/4$. The proof of Theorem 3.3.3 of Henry [4] applies, yielding existence of a unique solution $z(t)$ of the abstract equation satisfying, for this $\alpha$, and for all $\gamma < 1$,

$$z \in C([0,T),X^\alpha) \cap C^1((0,T),X^\gamma) \cap C((0,T),D(A)) . \tag{3.7}$$

(See Lemma 3.2.1 and Theorem 3.5.2 in [4] concerning differentiability of the solution. For an alternative source, see Pazy [13].) Taking components, we find that

$$S \in C([0,T),C_b^1) \cap C^1((0,T),C_b^1) \cap C((0,T),D(-\Delta)),\, U \in C^1([0,T),C^1) ;$$

the latter follows by taking $t$ to zero in (3.5). Now, recover $v$ from

$$v(x,t) = -\int_x^1 [S(y,t) - G^*(U(y,t),y)]dy .$$

It follows that

$$v \in C([0,T),C^2) \cap C^1((0,T),C^2), \qquad v_t = S_x \in C([0,T),C),$$

the system (3.1) is satisfied for $t \geq 0$, and $v(x,0) = v_0(x)$.

To prove the assertion regarding blow-up if global existence fails, we apply Henry's Theorem 3.3.4 to conclude that if the maximal $T$ is finite, then the $X^\alpha$ – norm of the solution $(S,U)$ of (3.5) must blow-up as $t \to T-$. But then it follows that the sup norm of $(S,U)$ must blow up, because otherwise, the nonlinear terms $f(z)$ in the abstract equation satisfy a linear growth estimate $\|f(z)\|_X \leq K(1 + \|z\|_{X^\alpha})$, and global existence follows from Corolla-y 3.3.5 in [4]. Now the assertion in (3.4) follows from the blow-up of the sup norm of $(S,U)$. ∎

Strong "semi-classical" solutions with discontinuous stress components may be obtained by a different choice of function spaces in the proof above. The following local existence theorem yields solutions of (3.1) in which $U$ and $v_x$ may be discontinuous in $x$, but $S_x$ and $v_t$ are continuous, and $U$ is $C^1$ as a function of $t$ for every $x$. Thus all derivatives in (3.1) may be interpreted in a classical sense as long as the equation is kept in conservation form. A result of this type was obtained by Pego in [14]; we follow a similar line of argument. In what follows $W^{1,\infty}$ is the space of Lipschitz continuous functions on $[0,1]$, and $W^{2,\infty}$ is the space of functions with Lipschitz continuous first derivatives.

**Theorem 3.2** (Local strong solutions): *Assume that $\partial G^*/\partial U$ and $H^*$ are locally Lipschitz functions. Assume that $v_o(1) = 0$, and that with $S_o(x) = v_{ox}(x) + G^*(U_o(x),x)$, we have $S_o \in H^1[0,1]$ with $S_o(0) = 0$, and that $U_o \in L^\infty[0,1]^N$. Then for some $T > 0$ a unique solution of* (3.5)–(3.6) *exists with*

$$S \in C([0,T),H^1) \cap C((0,T),W^{2,\infty}) \cap C^1((0,T),H^s) \quad \textit{for all} \quad s < 2,$$
$$U \in C^1([0,T),(L^\infty)^N).$$

*Moreover, there exists a unique solution of (3.1)-(3.3) with*

$$v \in C([0,T),W^{1,\infty}) \cap C^1((0,T),W^{1,\infty}).$$

*If $T$ is maximal but finite, then (3.4) holds. Also, given a bounded representative $U_{o^*}$ of the equivalence class $U_o$, there exists $T_* > 0$ and a unique bounded measurable function $U_*(x,t)$, representing $U(\cdot,t)$ for each $t$, such that $t \to U_*(x,t)$ is $C^1$ on $[0,T_*)$ for each $x$, and, identifying the equivalence class $S$ with its unique continuous representative, $(S,U_*)$ is a strong solution of (3.5).*

*Proof of Theorem 3.2.* The main part of the proof is a slight modification of the proof of Theorem 3.1. Now, the Banach space $X = L^2[0,1] \times L^\infty[0,1]^N$. One may show that with $\alpha = 1/2$, $D((-\Delta)^\alpha) = H^1 \cap \{S \mid \; S(0) = 0\}$, and this space is continuously embedded in $C[0,1]$. One obtains a solution $z(t)$ to the abstract equation with the properties in (3.7) for $\alpha = 1/2$, with $X^\alpha = D((-\Delta)^\alpha) \times L^\infty[0,1]^N$. The regularity properties for the solution follow by taking components and using the equations satisfied by $U_t$ and $S_{xx}$. The assertions regarding blow-up follow as before.

To establish the existence of the 'classical' solution $U_*$, we argue as follows. Identify the equivalence class $S$ with its unique continuous representative, and for each fixed $x$, let $U_*(x,t)$ be the unique solution of the ODE $dU_*/dt = H(U_*, S, x)$ with initial value $U_{o*}(x)$. Solutions exist on some interval $[0, T_*)$, $T_* \leq T$, with $T_*$ indenpendent of $x$, but the maximal $T_*$ may in principle depend on $U_{o*}$ and on how $H$ depends on $x$. Considering $U_t = H(U, S, x)$ as an ODE on $(L^\infty)^N$, it follows from uniqueness that $U_*$ agrees with $U$ for almost all $x$, for all $t \in [0, T_*)$. ∎

In the rest of this paper, $U$ will usually stand for a 'classical' solution $U_*$. We remark that as a simple corollary of the continuous dependence of solutions of $dU_*/dt = H(U_*, S, x)$ on initial data (or final data), it follows that *the solution $U_*$ is continuous at $(x_o, t_o)$ if and only if $U_{o*}$ is continuous at $x_o$.* In particular, discontinuities in the initial data for $U$ are preserved. In general, the spatial smoothness of the solution of (3.1)–(3.3) may be limited. However, so long as $G^*$ and $H^*$ are smooth, the solution is smooth as a function of time: The following proposition is a straightforward application of Corollary 3.4.6 in [4]. The hypotheses guarantee that $G$ and $H$ are $C^k$ functions of $(S, U)$, hence the nonlinearity $f(z)$ in the abstract equation is $C^r$ as a map from $X^\alpha$ to $X$. Henry's Corollary 3.4.6 in [4] asserts that the mapping $t \to z(t)$ is $C^r$ as a map on $(0, T)$ with values in $X^\alpha$.

**Proposition 3.3** (Temporal regularity): *Assume that partial derivatives of $G^*$ with respect to $U$ up to order $k+1$ are continuous, and partial derivatives of $H^*$ with respect to $U$ and $v_x$ up to order $k$ are continuous.*

a) *Suppose $r = k - 1 \geq 1$. Under the hypotheses of Theorem 3.1, we have*

$$S \in C^r((0,T), C_b^1), \, U \in C^r((0,T), (C^1)^N), \, v \in C^r((0,T), C^2)$$

b) *Suppose $r = k \geq 1$. Under the hypotheses of Theorem 3.2, we have*

$$S \in C^r((0,T), H^1), \, U \in C^r((0,T), (L^\infty)^N), \, v \in C^r((0,T), W^{1,\infty}).$$

We conclude this section by discussing criteria that imply the global existence of solutions of (3.1)–(3.3). Under the assumptions of Theorem 3.1 (or Theorem 3.2), the initial-boundary value problem (3.1)–(3.3) has a unique solution $(v(x,t), U(x,t))$ on $[0,1] \times [0,T)$ for some (maximal) $T > 0$. In view of (3.4) , if $v_x(x,t)$ and $U(x,t)$ are a-priori bounded on any interval $[0,1] \times [0, t_0]$, with $t_0 > 0$, then $T = \infty$ and the solution is defined for all times. In light of (2.16) and (2.17) this is the case for the problem (1.1)–(1.5).

**Corollary 3.4.** *Under the hypotheses stated in the Introduction, the initial-boundary value problem (1.1)–(1.5) has a unique solution existing globally in time.*

For a class of problems describing viscoelastic fluid flows, including (1.11) and others considered in the Appendix, the function $H^*$ in (3.1) has linear growth in $v_x$. For certain of these models it can be shown that $U$ satisfies an a priori $L^\infty$ estimate on $[0,1] \times [0, t_0]$ for any $t_0 > 0$ (cf. Appendix), and this suffices to guarantee that solutions are globally defined.

**Theorem 3.5** (Global Existence). *Let the assumptions of Theorem 3.1 (or Theorem 3.2) be satisfied and let*

$$H^*(U, v_x, x) = H_1(U, x) v_x + H_2(U, x). \tag{3.8}$$

*Let $(v(x,t), U(x,t))$ be a solution of (3.1)-(3.3) as in Theorem 3.1 (or Theorem 3.2) defined on $[0,1] \times [0,T)$, for some $T > 0$. If $T$ is the maximal time of existence, but $T$ is finite, then*

$$\limsup_{t \to T^-} [\sup_{x \in [0,1]} |U(x,t)|] = \infty\,. \tag{3.9}$$

*Proof of Theorem 3.5.* Let $(v(x,t), U(x,t))$ be a solution of (3.1)-(3.3) on $[0,1] \times [0,T)$, with $T$ maximal, and assume that (3.9) is violated; that is, there is a positive constant $K_1$ such that

$$|U(x,t)| \leq K_1 \quad \text{for} \quad (x,t) \in [0,1] \times [0,T)\,. \tag{3.10}$$

In view of (3.1) and (3.4), it suffices to show that

$$|S(x,t)| \leq K_2 \quad \text{for} \quad (x,t) \in [0,1] \times [0,T)\,, \tag{3.11}$$

where $K_2$ is some finite constant.

Indeed, by (3.1) and (3.8), $S(x,t)$ satisfies a linear parabolic equation of the form

$$S_t = S_{xx} + a(U,x)S + b(U,x) \tag{3.12}$$

with boundary conditions (3.6). The functions $a$ and $b$ are determined in terms of $G^*$, $H_1$ and $H_2$. We multiply (3.12) by $p|S|^{p-2}S$, $p \geq 2$, integrate by parts over $[0,1] \times [0,t]$, where $t < T$, and use (3.9) to arrive at

$$\int_0^1 |S(x,t)|^p dx \leq \int_0^1 |S_0(x)|^p dx + K_3 p \int_0^t \int_0^1 (1 + |S(x,\tau)|^p) dx d\tau\,. \tag{3.13}$$

Integrating (3.13) and taking the $1/p$ power, yields

$$(\int_0^1 |S(x,t)|^p dx)^{1/p} \leq [\int_0^1 |S_0(x)|^p dx + K_3 p T]^{1/p} e^{K_3 t}\,;$$

therefore, letting $p \to \infty$, we obtain (3.11). ∎

## 4. Convergence to Equilibria

Let $(v(x,t), \sigma(x,t))$ be a classical solution of (1.1)–(1.5) defined on $[0,1] \times [0,\infty)$. The existence of such solutions was discussed in Section 3. Here, we study the asymptotic behavior of $(v,\sigma)$ as $t \to \infty$.

The first lemma indicates that $S = \sigma + v_x + f\,x$ converges to its equilibrium value.

**Lemma 4.1.** *Under the assumptions of Theorem 3.1,*

$$\lim_{t\to\infty} S(x,t) = 0\,, \tag{4.1}$$

*uniformly for* $x \in [0,1]$.

*Proof.* The proof is a consequence of the a priori estimates

$$\int_0^\infty \int_0^1 S_t^2\, dx\, d\tau \le C\,, \tag{4.2}$$

$$\int_0^\infty \int_0^1 S^2\, dx\, d\tau \le C\,, \tag{4.3}$$

$$\int_0^1 S_x^2(x,t)\,dx \le C\,,\ 0 \le t < \infty\,, \tag{4.4}$$

where $C$ is a positive constant depending only on data. Once these have been established, the identity

$$\begin{aligned}\int_0^1 S^2(x,t)dx &= \int_{t-1}^t \int_0^1 S^2(x,\tau)dx\,d\tau \\ &+ 2\int_{t-1}^t \int_\tau^t \int_0^1 S(x,\eta)S_t(x,\eta)\,dx\,d\eta\,d\tau\,,\end{aligned} \tag{4.5}$$

together with (4.2) and (4.3), readily yield

$$\lim_{t\to\infty} \int_0^1 S^2(x,t)dx = 0\,. \tag{4.6}$$

Then (4.1) follows from (4.4), (4.6) and the inequality

$$S^2(x,t) \le 2\left(\int_0^1 S^2(x,t)dx\right)^{1/2}\left(\int_0^1 S_x^2(x,t)dx\right)^{1/2}. \tag{4.7}$$

It remains to prove (4.2)–(4.4). By (1.1), (4.4) is contained in (2.7). Also, (4.3) follows from (2.8) and (2.7). To prove (4.2), observe that $\sigma$ satisfies the equation

$$\sigma_{tt} + \sigma_t = g'(v_x)v_{xt} \tag{4.8}$$

whence, one derives the identity

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \sigma_t^2(x,t)dx + \int_0^1 \sigma_t^2(x,t)dx \\ = \int_0^1 g'(v_x(x,t))v_{xt}(x,t)\sigma_t(x,t)dx\,. \tag{4.9}$$

Using (2.7) and (2.17), (4.9) implies

$$\int_0^\infty \int_0^1 \sigma_t^2 dxdt \leq C\,. \tag{4.10}$$

By (1.3),

$$S_t = v_{xt} + \sigma_t\,. \tag{4.11}$$

Combining (4.11) with (4.10) and (2.7) we arrive at (4.2). ∎

Use of (4.1) enables us to identify the limiting behavior of solutions of (2.11) as $t \to \infty$. The following result is analogous to Lemma 5.5 in Pego [14].

**Lemma 4.2.** *Let $s(x,\bullet) \in C^1[0,\infty)$ be the solution of* (2.11), *where $S(x,\bullet)$ is continuous and satisfies* (4.1), $0 \leq x \leq 1$. *Then $s(x,\bullet)$ converges to $s^\infty(x)$ as $t \to \infty$ and $s^\infty(x)$ satisfies*

$$s^\infty(x) + g(s^\infty(x)) = f\,x \quad , \quad 0 \leq x \leq 1\,. \tag{4.12}$$

*Proof.* Let $x \in [0,1]$ be fixed. Set $\underline{s}(x) = \liminf_{t\to\infty} s(x,t)$ and $\overline{s}(x) = \limsup_{t\to\infty} s(x,t)$. By virtue of (2.16), $-\infty < \underline{s}(x) \leq \overline{s}(x) < \infty$.

Assume that $\underline{s}(x) < \overline{s}(x)$ and choose $s_0 \in (\underline{s}(x), \overline{s}(x))$. Then, there are sequences $\{t_n^-\}$, $\{t_n^+\}$ tending to infinity as $n \to \infty$, such that

$$s_t(x,t_n^+) \geq 0,\; s_t(x,t_n^-) \leq 0,\; s(x,t_n^\pm) = s_0 \tag{4.13}$$

and, by (2.11),

$$s_t(x,t_n^\pm) + s_0 + g(s_0 - S(x,t_n^\pm)) = f\,x\,. \tag{4.14}$$

Let $n \to \infty$ in (4.14) and use (4.13) and (4.1) to conclude

$$s_0 + g(s_0) = f\,x\,. \tag{4.15}$$

Thus, if $\underline{s}(x) < \overline{s}(x)$ any $s_0 \in (\underline{s}(x), \overline{s}(x))$ satisfies (4.15). Since the solution set of (4.15) is not connected, this is impossible. Hence $\underline{s}(x) = \overline{s}(x) =: s^\infty(x)$.

Next, we claim that $s^\infty(x) = \lim_{t\to\infty} s(x,t)$ satisfies (4.12). Indeed, given a fixed $x$ in $[0,1]$, there are sequences $\{t_n\}$ and $\{\tau_n\}$, $t_n \leq \tau_n \leq t_n + 1$ and $t_n \to \infty$, such that $|s_t(x,\tau_n)| = |s(x,t_n+1) - s(x,t_n)| \leq 1/n$. Evaluating (2.11) at $(x,\tau_n)$ and letting $n \to \infty$, we deduce that $s^\infty(x)$ satisfies (4.12) for any $0 \leq x \leq 1$. ∎

Let $(v(x,t), \sigma(x,t))$ be a classical solution of (1.1)–(1.5) on $[0,1] \times [0,\infty)$. Recalling the definition of $s$ in (2.10), Lemma 4.2 implies

$$\sigma^\infty(x) = \lim_{t\to\infty} \sigma(x,t) = s^\infty(x) - f\,x\,. \tag{4.16}$$

Also, combining (1.3), (4.1) and (4.16),

$$v_x^\infty(x) := \lim_{t\to\infty} v_x(x,t) = \lim_{t\to\infty}(S(x,t) - s(x,t)) = -s^\infty(x) \tag{4.17}$$

and

$$S^\infty(x) = v_x^\infty(x) + \sigma^\infty(x) + f\,x = 0\,. \tag{4.18}$$

Finally, noting that by (1.4)

$$v(x,t) = -\int_x^1 v_x(x,t)dx\,, \tag{4.19}$$

$v^\infty(x)$ is Lipschitz continuous and satisfies

$$v^\infty(x) := \lim_{t\to\infty} v(x,t) = \int_x^1 s^\infty(\xi)d\xi\,. \tag{4.20}$$

We conclude that any solution of (1.1)–(1.5) converges to one of the steady states. If $0 \le f < m$, then there is a unique smooth steady state which is the asymptotic limit of any solution. However, if $m < f$, then there are multiple steady states and thus a multitude of possible asymptotic limits. In Sec. 5, we identify stable steady states.

Observe that in case $w(\xi)$ is monotone the above arguments yield that every solution converges to the unique steady state. Moreover, the above results can be routinely generalized to the case that the function $w(\xi)$ has multiple loops but the graph of $w$ has no horizontal segments.

## 5. Stability of Steady States

The scope of this Section is to study the stability of velocity profiles with kinks. To fix ideas, let $(\overline{v}(x), \overline{\sigma}(x))$ be a steady state of (1.1)–(1.4) such that $\overline{v}(x)$ has a finite number of kinks located at the points $x_1, \ldots, x_n$ in $(0,1)$; accordingly, $\overline{v}_x(x)$ and $\overline{\sigma}(x)$ have a finite number of jump discontinuities at the same points. Recall that, if we set $\overline{u}(x) = -\overline{v}_x(x)$,

$$w(\overline{u}(x)) = f x\,,\; x \in [0,1],\; x \neq x_1, \ldots, x_n \tag{5.1}$$

and $\overline{\sigma}(x) = g(-\overline{u}(x))$.

Given any smooth initial data $(v_0(x), \sigma_0(x))$, there is a unique smooth solution $(v(x,t), \sigma(x,t))$ of (1.1)–(1.5). As $t \to \infty$, $(v(x,t), \sigma(x,t))$ converges to one of the steady states, not a-priori identifiable. We now restrict attention to initial data $(v_0(x), \sigma_0(x))$ such that the values of $(v_{0x}(x), \sigma_0(x))$ are close to $(\overline{v}_x(x), \overline{\sigma}(x))$ except on the union $\mathcal{U}$ of small subintervals centered around the points $x_1, \ldots, x_n$. $\mathcal{U}$ can be thought of as the location of transition layers separating the smooth branches of the steady state. Roughly speaking, it turns out that the steady state is "stable" under *smooth perturbations* that are close in energy, provided $(\overline{v}(x), \overline{\sigma}(x))$ takes values in the monotone increasing parts of $w(\xi)$. More precisely:

**Theorem 5.1.** *Let $(\overline{v}(x), \overline{\sigma}(x))$ be a steady state solution as described above and satisfying*

$$w'(\overline{v}_x(x)) \geq c_0 > 0\,,\; x \in [0,1], \qquad x \neq x_1, \ldots, x_n \tag{5.2}$$

*for some positive constant $c_0$. If the measure of $\mathcal{U}$* is sufficiently small, there is a positive constant $\delta_0$ *depending on $\mathcal{U}$ such that, if $\delta < \delta_0$, then for any initial data $(v_0(x), \sigma_0(x))$ satisfying*

$$\sup_{0 \leq x \leq 1} |S_0(x)| < \delta, \tag{5.3}$$

$$\int_0^1 v_t^2(x,0)dx < \frac{1}{2}\delta^2 \tag{5.4}$$

*and*

$$|v_{0x}(x) - \overline{v}_x(x)| < \delta \quad , \quad x \in [0,1] \setminus \mathcal{U} \tag{5.5}$$

*the corresponding solution $(v(x,t), \sigma(x,t))$ approaches the steady state $(\overline{v}(x), \overline{\sigma}(x))$ as $t \to \infty$, in the sense,*

$$v_x(x,t) \to \overline{v}_x(x)\,, \tag{5.6}$$

$$\sigma(x,t) \to \overline{\sigma}(x)\,, \tag{5.7}$$

*for all $x \in [0,1] \setminus \mathcal{U}$.*

*Proof*: To simplify the exposition, we prove the theorem for the case that $\overline{u}(x) = -\overline{v}_x(x)$ has one single jump discontinuity located at $x_0$, $m < fx_0 < M$, and for $\mathcal{U} = (x_0 - \varepsilon, x_0 + \varepsilon)$ for some small $\varepsilon$. Minor modifications are needed to account for the general case.

The proof is based on exploiting the energy identity (2.6), which, upon setting $u(x,t) = -v_x(x,t)$ yields the inequality

$$\begin{aligned}\frac{1}{2}\int_0^1 v_t^2(x,t)dx + \int_0^1 [(W(u(x,t)) - xfu(x,t)) - \Phi(x)]dx \\ \leq \frac{1}{2}\int_0^1 v_t^2(x,0)dx + \int_0^1 [(W(u_0(x)) - xfu_0(x) - \Phi(x)]dx.\end{aligned} \tag{5.8}$$

The function $\Phi(x)$ is associated with the particular choice of $\overline{u}(x)$, it identifies a basin of attraction of $\overline{u}(x)$ and is constructed below.

For each fixed $x$, the function $\overline{u}(x)$ occupies the bottom of one of the wells of the potential $W(u) - xfu$, in the left well if $x < x_0$ and in the right well if $x > x_0$.
Let

$$e(x) := W(\overline{u}(x)) - xf\overline{u}(x) \tag{5.9}$$

be the corresponding value of the potential at this $x$, and let

$$E := \inf_{\xi \in R}\inf_{0 \leq x \leq 1}(W(\xi) - xf\xi) \geq -\frac{1}{2}f^2 \tag{5.10}$$

be a lower bound of the potential valid for all $x \in [0,1]$. The function $\Phi(x)$ is now defined by

$$\Phi(x) := \begin{cases} e(x) & \text{if } |x - x_0| \geq \varepsilon \\ E & \text{if } |x - x_0| < \varepsilon\,. \end{cases} \tag{5.11}$$

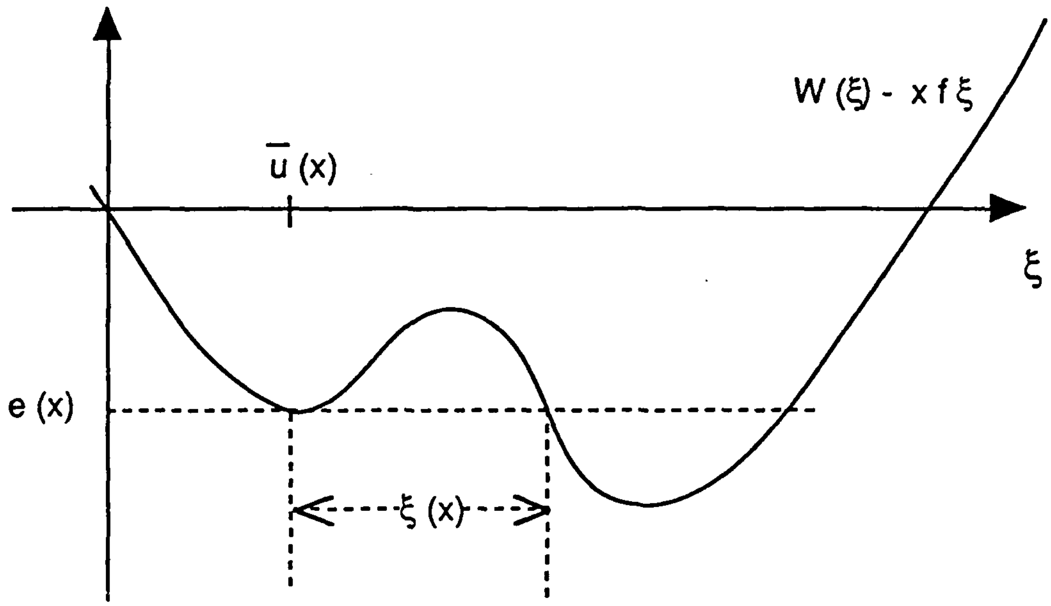


**Fig. 4:** Graphs of $W(\xi)-xf\xi$, $e(x)$ and $\xi(x)$ for $x$ fixed, $x < x_0-\varepsilon$ and $fx$ above the Maxwell line.

Moreover, set

$$\xi(x) := \begin{cases} sup\{\zeta \in R : W(\xi) - xf\xi \geq e(x) \quad \text{for } -\infty < \xi < \zeta\}, & x < x_0 \\ inf\{\zeta \in R : W(\xi) - xf\xi \geq e(x) \quad \text{for } \zeta < \xi < \infty\}, & x > x_0 \end{cases} \tag{5.12}$$

and observe that $\xi(x)$ equals $+\infty$ or $-\infty$ in case $\overline{u}(x)$ is positioned at the lower of the two wells, while $\xi(x)$ is finite otherwise. The functions $W(\xi) - xf\xi$, $e(x)$ and $\xi(x)$ used in the construction of $\Phi(x)$ are shown in Fig. 4 for some $x$ fixed, with $x < x_0 - \varepsilon$ and $fx$ above the Maxwell line.

The following lemma identifies the region of attraction of the steady state solution.

**Lemma 5.2.** *There is a positive constant $\rho_0$ such that, given $\rho < \rho_0$, there are positive functions $r_m(\rho)$ and $r_M(\rho)$, with $r_m(\rho) < r_M(\rho)$ and the property: For any $r \in (r_m(\rho), r_M(\rho))$, if*

$$\sup_{0\leq x\leq 1} |S_0(x)| < \rho\,, \tag{5.13}$$

$$\mathcal{E}(0) := \frac{1}{2}\int_0^1 v_t^2(x,0)dx + \int_0^1 [W(u_0(x)) - xfu_0(x) - \Phi(x)]dx < \frac{1}{2}\rho^2 \tag{5.14}$$

*and*

$$|u_0(x) - \overline{u}(x)| < r - 2\rho, \qquad x \in [0,1]\setminus\mathcal{U}\,, \tag{5.15}$$

*then*

$$\sup_{0\leq x\leq 1} |S(x,t)| < \rho \tag{5.16}$$

*and*

$$|u(x,t) - \overline{u}(x)| < r, \qquad x \in [0,1]\setminus\mathcal{U}\,, \tag{5.17}$$

*for any* $t \in [0,\infty)$.

*Proof.* Let $t^* = sup\{t_0 \in R^+ : \sup_{0\leq x\leq 1} |S(x,t)| < \rho \quad for \quad 0 \leq t < t_0\}$ be the first time at which (5.16) is violated. By (5.13), $t^* > 0$. We claim that, if $\rho$ is sufficiently small, $t^* = +\infty$. Suppose that $t^* < +\infty$. Then for any $t_0 < t^*$,

$$\sup_{0\leq x\leq 1} |S(x,t)| < \rho, \qquad 0 \leq t \leq t_0\,, \tag{5.18}$$

while,

$$\sup_{0\leq x\leq 1} |S(x,t^*)| = \rho\,. \tag{5.19}$$

We now refer to the discussion in Section 2 on invariant intervals of uniform length for the parametric family of ode's (2.11). Set $\overline{s}(x) = \overline{u}(x) = -\overline{v}_x(x)$. Let $\rho_0, R_0$ and, for $\rho < \rho_0$, let $r_1(\rho)$ and $r_2(\rho) = R_0 - r_1(\rho)$ be as in Corollary 2.2; also recall that $r_1(\rho) \to 0$ as $\rho \to 0$. Set $r_m(\rho) := max\{2\rho, r_1(\rho) + \rho\}$ and $r_M(\rho) := min\{R_0, R_0 - r_1(\rho) + \rho\}$. Taking $\rho_0$ even smaller if needed, it follows that $r_m(\rho) < r_M(\rho) \leq R_0$ for all $\rho < \rho_0$. Fix $r \in (r_m(\rho), r_M(\rho))$. Then equations (1.6) and (2.10), (5.13) and (5.15) imply

$$|s_0(x) - \overline{s}(x)| < r - \rho, \qquad x \in [0,1] \setminus \mathcal{U}\,. \tag{5.20}$$

Moreover, Corollary 2.2, together with (1.3), (2.10) and (5.18), yields

$$|u(x,t) - \overline{u}(x)| \leq |s(x,t) - \overline{s}(x)| + |S(x,t)| < r < R_0\,, \qquad x \in [0,1] \setminus \mathcal{U}\,. \tag{5.21}$$

By virtue of the definition of $R_0$ and (5.12), $R_0 \leq \xi(x)$, for all $0 \leq x \leq 1, x \neq x_0$. Thus, (5.11) and (5.21) guarantee that

$$W(u(x,t)) - xfu(x,t) \geq \Phi(x), \quad 0 \leq x \leq 1, 0 \leq t \leq t_0\,. \tag{5.22}$$

To conclude the proof, combine (5.8) with (5.22) to obtain

$$\frac{1}{2}\int_0^1 v_t^2(x,t)dx \leq \mathcal{E}(0), \qquad 0 \leq t \leq t_0\,. \tag{5.23}$$

Then (5.23) and (2.8) together imply that

$$\sup_{0\leq x\leq 1} |S(x,t_0)| \leq \sqrt{2\mathcal{E}(0)} \tag{5.24}$$

for any $t_0 < t^*$. But (5.24) and (5.14) together contradict (5.19). Thus $t^* = \infty$ and (5.16) holds in $[0,\infty)$. Moreover, (5.21) yields (5.17). ∎

A remark is appropriate at this point to guarantee that the assumptions (5.13)–(5.15) are not vacuous but follow from a simpler set of assumptions such as (5.3)–(5.5). Recall that the constants $\rho_0$ and $R_0$ required in Lemma 5.2 depend on $\overline{u}(x)$, and the form of $w(\xi)$

but not on the width $\varepsilon$ of the layer in the initial data. If $\delta \leq min\{\rho, r-2\rho\}$, then (5.3)–(5.5) imply (5.13) and (5.15). They will also imply (5.14) provided

$$\int_0^1 [W(u_0(x)) - xf(u_0(x)) - \Phi(x)]dx < \frac{1}{4}\rho^2 . \tag{5.25}$$

To satisfy (5.25) two things are needed: that $\delta$ be chosen small enough and that $\varepsilon$ not exceed a certain value. Moreover, the smaller $\varepsilon$ is chosen, the larger $\delta$ can be taken.

Lemma 5.2 guarantees that steady states $(\overline{v}, \overline{\sigma})$ satisfying (5.2) are stable under perturbations that are close to $(\overline{v}, \overline{\sigma})$ in energy. Actually more is true. It follows from Lemma 4.2 that $v_x(x,t) \to v_x^\infty(x)$ and $\sigma(x,t) \to \sigma^\infty(x)$ for a.e. $x \in [0,1]$ as $t \to \infty$. Using (5.17),

$$|v_x(x,t) - \overline{v}_x(x)| = |u(x,t) - \overline{u}(x)| < r, \quad x \in [0,1] \setminus \mathcal{U} , \tag{5.26}$$

and the fact that the only solution of (5.1) in $[\overline{u}(x) - r, \overline{u}(x) + r]$, for $r < R_0$, is $\overline{u}(x)$ itself, we conclude that $v_x^\infty(x) = \overline{v}_x(x)$ and $\sigma^\infty(x) = \overline{\sigma}(x)$ for $x \in [0,1] \setminus \mathcal{U}$. ∎

## Appendix

In this appendix, we use the results of Sec. 3 to obtain global existence results for equations governing motions of incompressible viscoelastic fluids in simple planar shear. We consider a class of one-dimensional models popular in polymer rheology based on differential constitutive laws which satisfy *the principle of frame indifference*; we refer to [16] for a general formulation.

**1. Johnson-Segalman / Oldroyd Models** For a class of models with $m$ - relaxation times, derived from three-dimensional differential constitutive relations due to Johnson & Segalman [6] (with exponential memory functions) and Oldroyd [12], the equations of motion take the following form in one space dimension (see [10] for a detailed formulation):

$$\begin{aligned} &\rho v_t = S_x \ , \\ &S = \sum_{j=1}^{m} \sigma_j + \eta v_x + f x \ , \\ &\sigma_{jt} + \lambda_j \sigma_j = (Z_j + \mu_j)\, v_x \ , \\ &Z_{jt} + \lambda_j Z_j = -\left(1 - a_j^2\right) \sigma_j v_x \, , \quad j = 1, ..., m \, , \end{aligned} \tag{A.1}$$

where $\rho$ is the fluid density, $v$ is the fluid velocity, $\sigma_j$ represents a contribution to the total shear stress due to the polymer, $\eta v_x$ is the Newtonian contribution to the total shear stress, and $Z_j$ represents a contribution to the principal normal stress difference; $\eta, f, \lambda_j, \mu_j, a_j$ are positive constants where $-1 < a_j < 1$; $\lambda_j$ are relaxation rates, $\mu_j$ are elastic shear moduli, and $a_j$ are slip parameters.

The initial-boundary value problem governing Poiseuille shear flow between parallel plates at $x = \pm 1$ with the flow symmetric about the centerline, and driven by a constant

pressure gradient consists of the system (A.1) on $[0,1] \times [0,\infty)$, coupled with boundary conditions (1.4), and initial conditions

$$v(x,0) = v_0(x),\ \sigma_j(x,0) = \sigma_{j0}(x),\ Z_j(x,0) = Z_{j0}(x)\,, \tag{A.2}$$

on $0 \le x \le 1, j = 1,...,m$; it is assumed that the compatibility conditions $v_0(1) = 0$, $v_0'(0) = 0$, and $\sigma_{j0}(0) = 0, j = 1,...,m$, are satisfied. In what follows, we assume that the initial data satisfy the hypothesis of Theorem 3.5. We note that when $\eta = 0$, it is readily shown that (A.1) is hyperbolic if $\sum_{j=1}^{m}(Z_j + \mu_j) \ge 0$, with wave speeds $\pm\{\frac{1}{\rho}\sum_{j=1}^{m}(Z_j+\mu_j)\}^{1/2}$ and 0 (repeated $m+1$ times).

In the case of a single relaxation time, we omit the subscript $j$, and we non-dimensionalize the variables by scaling distance by $h$, time by $\lambda^{-1}$, and stress by $\mu$. Furthermore, if we replace $\sigma$, $v$, and $f$ by $\hat{\sigma} := (1-a^2)^{1/2}\sigma$, $\hat{v} := (1-a^2)^{1/2}v$, and $\hat{f} := (1-a^2)^{1/2}f$, respectively, then the parameter $a$ disappears from Eqs. (A.1). Since no confusion will arise, we omit the caret. There are two essential dimensionless parameters: $\alpha := \rho h^2\lambda^2/\mu$, a ratio of Reynolds number to Deborah number, and $\varepsilon := \eta\lambda/\mu$, a ratio of viscosities. The special case of (A.1) with $j = 1$ leads to the initial–boundary-value problem consisting of the system (1.10) on $[0,1]\times[0,\infty)$, with boundary conditions (1.4) and initial conditions (A.2), $j = 1$.

To establish a global existence result for the system (A.1), with boundary conditions (1.4) and initial conditions (A.2), we observe that (A.1) is endowed with the identities

$$\begin{aligned}&\frac{\partial}{\partial t}[(1-a_j^2)\sigma_j^2 + (Z_j+\mu_j)^2] \\ &\quad = -2\lambda_j[(1-a_j^2)\sigma_j^2 + (Z_j + \frac{\mu_j}{2})^2 - \frac{\mu_j^2}{4}]\,,\end{aligned} \tag{A.3}$$

for $j = 1,...,m$, and any $x \in [0,1]$. For each $j$, the right side of (A.3) is strictly negative in the exterior of the ellipse

$$\Gamma_j := \{(\sigma_j, Z_j) : (1-a_j^2)\sigma_j^2 + (Z_j + \frac{\mu_j}{2})^2 = \frac{\mu_j^2}{4}\}\,. \tag{A.4}$$

Consider the closed sets

$$\Omega_j := \{(\sigma_j, Z_j) : (1-a_j^2)\sigma_j^2 + (Z_j+\mu_j)^2 \le C_j^2\}\,, \tag{A.5}$$

$j = 1,...,m$, where $C_j \ge 0$ are constants. If $C_j > \mu_j, , j = 1,...,m$, the ellipses bounded by $\Gamma_j$ are (properly) contained in the sets $\Omega_j$ for each $j$. Letting $U := (\sigma_1, Z_1, ..., \sigma_m, Z_m)$ and defining $\Omega := \bigcup_{j=1}^{m}\Omega_j$, the above construction implies that if $C_j \ge \mu_j (j = 1,...,m)$, $U \in L^\infty([0,1]\times[0,T])$ for every $T > 0$, where the a priori bound depends only on the initial data and not on $T$. Thus Theorem 3.5 yields that solutions of the initial-boundary value problem (A.1), (1.4), (A.2) exist globally in time.

Finally, we remark that the same approach can be applied to obtain global-time existence for shearing flows of non-Newtonian fluids governed by differential constitutive

relations due to Phan-Thien and Tanner [15] . In the case of one relaxation time, the relevant system in dimensionless form (compare with (1.10)) is

$$\begin{aligned} \alpha v_t &= \sigma_x + \varepsilon v_{xx} + f \ , \\ \sigma_t + e^{\delta Z}\sigma &= (Z+1)\, v_x \ , \\ Z_t + e^{\delta Z} Z &= -\sigma v_x \ , \end{aligned} \tag{A.6}$$

where $\alpha, \varepsilon, v, \sigma$ have the same meaning as in (1.10) and where the constant $\delta > 0$. To establish a global existence result for the initial-boundary value problem associated with (A.6), we use the same approach as for (A.1). For the application of Theorem 3.5, the identity for (A.6), corresponding to the identity (A.3) with $j = 1$, now takes the form

$$\frac{\partial}{\partial t}[\sigma^2 + (Z+1)^2] = -2e^{\delta Z}[\sigma^2 + (Z+\frac{1}{2})^2 - \frac{1}{4}] \, . \tag{A.7}$$

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